

Four-Index Equations for Gravitation and the Gravitational Energy-Momentum Tensor *

Zahid Zakir[†]

Centre for Theoretical Physics and Astrophysics,
P.O.Box 4412, Tashkent 700000 Uzbekistan

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Abstract

A new treatment of the gravitational energy on the basis of 4-index gravitational equations is reviewed. The gravitational energy for the Schwarzschild field is considered.

1 Introduction

A covariant physical characteristics of the gravitational field is the Riemann curvature tensor, and it is natural that the problems with the energy-momentum of gravitation can be solved if we can express the gravitational energy in terms of this tensor.

In the papers [1],[?] a new generalized 4-index version of the Einstein equations with the Riemann tensor has been formulated, and the local energy-momentum tensors for the system of gravitation field and matter, linearly depending on the curvature tensor, have been constructed as 4-index tensors.

In the present paper some consequences of this treatment, including the calculation of the gravitational energy for a mass point, will be presented.

2 Four-index equations for the gravitational field

In the standard Einstein-Gilbert gravitational action one can add to the Ricci tensor or to the Riemann tensor arbitrary functions (tensors) with zero contractions:

$$R = g^{km} R_{km} = \frac{1}{2} (g^{km} g^{il} - g^{im} g^{kl}) (R_{iklm} - \kappa V_{iklm}), \quad (1)$$

where $\kappa = 8\pi k$, and L_m is the matter Lagrangian, V_{iklm} has the same symmetry properties as R_{iklm} , and $g^{il} V_{iklm} = 0$.

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[†]E-mail: zahid@in.edu.uz

So, we can start from the new action function :

$$S = \frac{1}{2} \int d\Omega \sqrt{-g} \left[\frac{1}{2} (g^{km} g^{il} - g^{im} g^{kl}) \left(-\frac{1}{\kappa} R_{iklm} + V_{iklm} \right) + L_m \right], \quad (2)$$

which is fully equivalent to the Einstein-Gilbert action function Eq. (??). Then we obtain for the variation of the action function [1]:

$$\delta S = -\frac{1}{2} \int d\Omega \sqrt{-g} \delta g^{km} g^{il} (G_{iklm} - T_{iklm}) = 0, \quad (3)$$

where $T_{iklm} = V_{iklm} + T_{iklm}^{(m)}$, and:

$$G_{iklm} = \frac{1}{\kappa} \left[R_{iklm} - \frac{1}{(d-1)(d-2)} (g_{il} g_{km} - g_{im} g_{kl}) R \right], \quad (4)$$

$$\begin{aligned} T_{iklm}^{(m)} = & \frac{1}{(d-2)} (g_{km} T_{il} - g_{kl} T_{im} + g_{il} T_{km} - g_{im} T_{kl}) - \\ & - \frac{1}{(d-1)(d-2)} (g_{il} g_{km} - g_{im} g_{kl}) T. \end{aligned} \quad (5)$$

Here d is the spacetime dimensionality, and T_{iklm} has the same structure as the Riemann tensor having the representation:

$$\begin{aligned} R_{iklm} = & C_{iklm} + \frac{1}{(d-2)} (g_{km} R_{il} - g_{kl} R_{im} + g_{il} R_{km} - g_{im} R_{kl}) - \\ & - \frac{1}{(d-1)(d-2)} (g_{il} g_{km} - g_{im} g_{kl}) R, \end{aligned} \quad (6)$$

where C_{iklm} is the Weyl tensor with zero contractions $g^{il} C_{iklm} = g^{km} C_{iklm} = 0$.

Thus, we obtain the equations:

$$g^{il} (G_{iklm} - T_{iklm}) = 0. \quad (7)$$

In a general case the expression in the parenthesis is not equal to zero for the arbitrary V_{iklm} and we can not simply exclude the contractional factor g^{il} . However the tensor V_{iklm} has 10 independent components which is equal to the number of the Riemann tensor components in the vacuum ($T_{km}^{(m)} = 0$) where it is reduced to the Weyl tensor. Therefore, if in this case we choose the V_{iklm} as equal to:

$$\frac{1}{\kappa} G_{iklm} = \frac{1}{\kappa} C_{iklm} = V_{iklm}, \quad (8)$$

the equations hold identically for the solutions of the Einstein equations. Thus, we may write the 4-index equations for the gravitational field as [1]:

$$G_{iklm} = T_{iklm}. \quad (9)$$

We see that V_{iklm} can be considered as the 4-index energy-momentum density tensor for the gravitational field. Although its 2-index contraction vanish, in the 4-index form it

allows one to determine a nonzero, local and positive defined energy-momentum tensor for the gravitational field.

The tensors G_{iklm} and T_{iklm} have the symmetry properties of the Riemann tensor and, therefore, we have 20 equations. The tensor G_{iklm} is a function of the metric tensor g_{ik} which has 6 independent components. The tensor $T_{iklm}^{(m)}$ has been combined from the ordinary energy-momentum tensor of the matter T_{ik} and it has 4 independent functions (the energy density ϵ and 3 components of the velocity). These 10 functions obey to 10 Einstein equations. The tensor V_{iklm} gives additional 10 independent components.

So, we have 20 equations for 20 independent functions. If we take solutions of the Einstein equations for the metrics and T_{ik} , then we have the additional 10 equations for 10 components of V_{iklm} . Therefore, the solutions of the Einstein equations exactly define all components of V_{iklm} and in the paper we can find V_{iklm} for the known standard metrics.

In the vacuum $T_{ik} = T = 0$, $R_{il} = R = 0$ and we have the equations Eq.(8). We see that in the vacuum the tensor V_{iklm} plays the role of the source for the empty spacetime curvature C_{iklm} .

The covariant derivatives of the 4-index tensors are also related as:

$$G_{klm;i}^i = T_{klm;i}^i. \quad (10)$$

In the case $d = 4$ we have:

$$G_{.klm;i}^i = T_{km;l} - T_{kl;m} - \frac{1}{3}(g_{km}T_{;l} - g_{kl}T_{;m}), \quad (11)$$

$$T_{klm;j}^{j(m)} = \frac{1}{2} \left[T_{km;l} - T_{kl;m} - \frac{1}{3}(g_{km}T_{;l} - g_{kl}T_{;m}) \right] = \frac{1}{2} G_{.klm;i}^i. \quad (12)$$

Then we obtain the relationship:

$$V_{klm;j}^j = G_{klm;j}^j - T_{klm;j}^{j(m)} = \frac{1}{2} G_{.klm;i}^i. \quad (13)$$

and, therefore,

$$V_{klm;j}^j = T_{klm;j}^{j(m)} \quad (14)$$

In the vacuum, therefore, there are local conservation laws:

$$G_{.klm;j}^j = V_{klm;j}^j = 0. \quad (15)$$

The integral energy-momentum tensor for the system of matter and gravitational field can be defined as:

$$P_{lm}^i = \int dS_k T_{.lm}^{ik}. \quad (16)$$

On the hypersurface $x^0 = \text{const}$ we have:

$$P_{.lm}^k = \int d^3x \sqrt{-g} T_{.lm}^{0k} = \int d^3x \sqrt{-g} (T_{.lm}^{(m)0k} + V_{.lm}^{0k}). \quad (17)$$

The energy-momentum vector for matter can be obtained as: $P^i = P_{.k}^{ik}$. Finally, the 3-index integral energy-momentum of the gravitational field can be defined as:

$$P_{.lm}^{(g)i} = \int d^3x \sqrt{-g} V_{.lm}^{0i}. \quad (18)$$

3 The gravitational energy for the Schwarzschild field

Let us consider the energy of the Schwarzschild field with the line element:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (19)$$

where $r_g = 2Gm$ is the gravitational radius, and the components of the metric are: $g_{22} = -r^2$, $g_{33} = -r^2 \sin^2 \vartheta$, and:

$$g_{00} = g_{11}^{-1} = 1 - \frac{r_g}{r}. \quad (20)$$

We calculate the energy-momentum tensor:

$$V_{lm}^{ik} = \frac{1}{\kappa} R_{lm}^{ik} \quad (21)$$

for this solution of the Einstein equations. Nonzero components of the $V_{iklm} = R_{iklm}/\kappa$ with this metric are:

$$V_{0101} = \frac{r_g}{\kappa r^3} = -V(r)g_{00}g_{11}, \quad (22)$$

$$V_{0202} = -\frac{r_g(r - r_g)}{2\kappa r^2} = \frac{1}{2}V(r)g_{00}g_{22}, \quad (23)$$

$$V_{0303} = -\frac{r_g(r - r_g)}{2\kappa r^2} \sin^2 \vartheta = \frac{1}{2}V(r)g_{00}g_{33}, \quad (24)$$

$$V_{1212} = \frac{r_g}{2\kappa(r - r_g)} = \frac{1}{2}V(r)g_{11}g_{22}, \quad (25)$$

$$V_{1313} = \frac{r_g \sin^2 \vartheta}{2\kappa(r - r_g)} = \frac{1}{2}V(r)g_{11}g_{33}, \quad (26)$$

$$V_{2323} = -\frac{r_g r}{\kappa} \sin^2 \vartheta = -V(r)g_{22}g_{33}, \quad (27)$$

where:

$$V(r) = \frac{r_g}{\kappa r^3} = \frac{m}{4\pi r^3} = -\frac{m}{8\pi} \frac{\partial}{\partial r}(r^{-2}). \quad (28)$$

We see, that the 2-index contraction of this tensor vanishes:

$$V_{il} = g^{km} V_{iklm} = g_{il}[-V(r) + \frac{1}{2}V(r) + \frac{1}{2}V(r)] = 0.$$

The physical components of the gravitational energy-momentum tensor $V_{..lm}^{ik} = g^{ip}g^{kq}V_{pqlm}$ are:

$$V_{..01}^{01} = V_{..10}^{10} = V_{..23}^{23} = V_{..32}^{32} = -V(r), \quad (29)$$

$$V_{..02}^{02} = V_{..20}^{20} = V_{..03}^{03} = V_{..30}^{30} = V_{..12}^{12} = \quad (30)$$

$$= V_{..21}^{21} = V_{..13}^{13} = V_{..31}^{31} = \frac{1}{2}V(r). \quad (31)$$

They allow us to calculate one of components of integral gravitational energy-momentum of the static mass point as:

$${}_cP_{.01}^{(g)1} = \int dS_0 \sqrt{-g} V_{.01}^{01} = \int dS_0 \sqrt{-g} [-V(r)]. \quad (32)$$

The spatial volume integral can be represented as a spatial surface integral, and we obtain:

$${}_cP_{.01}^{(g)1} = \frac{m}{8\pi} \int dS_0 \sqrt{-g} \frac{\partial}{\partial r} (r^{-2}) = \frac{m}{8\pi} \int df_{0r} r^{-2} = \frac{1}{2} n_r m, \quad (33)$$

where $df_{0r} = n_r r^2 do$ is 2-dimensional surface element with the normal vector n_r along r .

References

- [1] Zakir Z. (1999) *New Equations for Gravitation with the Riemann Tensor and 4-Index Energy-Momentum Tensors for Gravitation and Matter*. (gr-qc/9905009), in "Zakir Z. (2003) *Structure of Space-Time and Matter*. CTPA, Tashkent".
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